As a provisional definition, *clifford algebra* over a field $k$ can be defined as

$$Cl_n(k) = k[x_1, \dots, x_n]/R$$

where,

$$R =< x_i^2 = -1, x_i x_j = -x_j x_i \forall i \neq j >$$

Easy to see that $Clifford\ Algebra$ has dimension $2^n$. $Cl_n(k)$ can also be thought of as

$$Cl_n(k) = TV/R$$

where $V = k^n$ as vector space, $TV$ is the tensor algebra, $R$ is same as above, except that $x_i$'s are standard basis for $V$. This observation leads to a more general definition of *clifford algebra*, where $V$ is a vector space equipped with a symmetric bilinear form

**Definition** Let $V$ be a vector space with symmetric biliear form $\beta$ and quadratic form $q = \beta(x, x)$. Then the *clifford algebra* over $V$ can be defined as

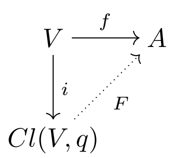
$$Cl(V, q) = TV/ < x \otimes x + q(x) >$$

**Remark** These are some of the properties that $CL(V, q)$ enjoys

(i) There is a natural inclusion of $i : V \hookrightarrow Cl(V, q)$.
(ii) If $q(x) = 0 \forall x \in V$ then $Cl(V, q) = \bigwedge(V)$ the exterior algebra
(iii) Let . denote the *clifford multiplication*( induced by the tensor product of $TV$), then

$$x.y + y.x = -2\beta(x, y)$$

(iv) **Universal Property :** Let $f : V \longrightarrow A$, where $A$ is a $k$-algebra, such that $f(x)^2 = -q(x)$, then there exists an unique map $F$ such that $f = F \circ i$, that is the following diagram commutes




(v) A map $\phi : (V, q) \longrightarrow (V', q')$, such that $q'(\phi(x)) = q(x)$, extends to a $k$-algebra homomorphism

$$\Phi : Cl(V, q) \longrightarrow Cl(V', q')$$

Thus the orthogonal group $\mathbb{O}(n)$ has an action on $Cl_n(\mathbb{R})$

**Proposition 0.1.** *$Cl(v, q)$ is a filtered algebra whose associate graded is $\bigwedge(V)$*

Before proving the theorem, recall the following definition

**Definition** If $A$ is a $k$-algebra then a filtration $(A, \mathfrak{F})$ of $A$ is sequence of subspaces

$$F_0 A \subset F_1 A \subset \cdots \subset A = \bigcup_r F_r A$$

The *associate graded* of $(A, \mathfrak{F})$ is defined as

$$Gr(A, \mathfrak{F}) = \bigoplus_r F_r A / F_{r-1} A$$

*Proof.* Let $\pi$ be the quotient map

$$\pi : TV \longrightarrow Cl(V, q)$$

Define, $G_r = V \otimes \ldots \otimes V$ ($r$ fold tensor product). Define filtration on $TV$ by setting

$$F_r TV = \bigoplus_{i \leq r} G_i$$

Define filtration $\pi_* F_r TV$ be the filtration on the clifford algebra. Note $x \otimes x = q(x) \in \pi_* F_0 Cl(V, q)$. Hence, in the associated graded $x \otimes x = 0$. On the other hand the relation $x_i x_j = -x_j x_i$ prevails in the associated graded. Hence the associate graded is isomorphic to $\bigwedge V$. □

**Remark** $Cl(V, q)$ is a $\mathbb{Z}/2$-graded algebra.

$$Cl^0(V, q) = \pi_* TV^{even}$$

$$Cl^1(V, q) = \pi_* TV^{odd}$$

**Definition** Recall,$S^{n-1} \subset V \hookrightarrow Cl(V, q)$. Define,

$$Pin(n) = S^{n-1} \subset Cl^{\times}(V, q)(units)$$

and

$$Spin(n) = S^{n-1} \cap Cl^0(V, q)$$

On $Cl(V, q)$ we have an involution map, which is induced by the involution on $TV$ given by,

$$\overline{x_1 \otimes \ldots \otimes x_r} = (-1)^r x_1 \otimes \ldots \otimes x_r$$

If $x \in S^{n-1} \subset Cl(V.q)$ then

$$x.\overline{x} = -\beta(x, x) = q(x) = 1$$

Let $v \in V \subset Cl(V, q)$ and $x \in Pin(n)$, then observe

$$q(-x.v.\overline{x}) = \beta(-x.v.\overline{x}, -x.v.\overline{x}) = x.(-q(v)).x = -q(v).(-q(x)) = q(v)$$

**Lemma 0.2.** *There exist short exact sequences*

$$1 \longrightarrow \{-1, +1\} \longrightarrow Pin(n) \xrightarrow{p} \mathbb{O}(n) \longrightarrow 1$$

*and*

$$1 \longrightarrow \{-1, +1\} \longrightarrow Spin(n) \xrightarrow{p} \mathbb{SO}(n) \longrightarrow 1$$

*where $p$ is the map which sends*

$$x \mapsto (v \mapsto -x.v.\overline{x)}$$

Let $k$ be a field. Recall, tensor product of $k$-algebras $A$ and $B$ is a $k$-algebra, denoted by $A \otimes_k B$ and multiplication is given by

$$(a \otimes b).(a' \otimes b') = aa' \otimes bb'$$

moreover if $M_j(A)$ denotes the set of all $j \times j$ matrices. Then we have the following isomorphism

$$M_j(A) \otimes M_k(B) \cong M_{jk}(A \otimes B)$$

Define

$$Cl_{p,q}(k) = Cl(k^{p+q}, x_1^2 + \ldots + x_p^2 - x_{p+1}^2 - \ldots x_{p+q}^2)$$

**Remark** If $k = \mathbb{C}$, then
$$Cl_{p,q}(\mathbb{C}) \cong Cl_{p+q,0}(\mathbb{C})$$
This follows from the fact that the quadratic forms $q_{p,q}(x) = x_1^2 + \ldots + x_p^2 - x_{p+1}^2 - \ldots x_{p+q}^2$ and $q_{p+q,0}(x) = x_1^2 + \cdots + x_{p+q}^2$ induces isomorphic innerproduct structure on $\mathbb{C}^{p+q}$ where the isomorphism sends
$$e_t \mapsto e_t : 0 \leq t \leq p$$
$$e_t \mapsto ie_t : p < t \leq p + q$$

**Theorem 0.3.** *If $k = \mathbb{C}$, then we have the following isomorphisms*

*(i)* $Cl_0(\mathbb{C}) = \mathbb{C}$
*(ii)* $Cl_1(\mathbb{C}) = \mathbb{C} \times \mathbb{C}$
*(iii)* $Cl_2(\mathbb{C}) = M_2(\mathbb{C})$
*(iv)* $Cl_{n+2}(\mathbb{C}) = M_2(Cl_n(\mathbb{C}))$

**Corollary 0.4.** *(Bott Periodicity) As a consequence of (iv) we have*
$$Cl_n(\mathbb{C}) = M_{2^n}(\mathbb{C})$$
*if $n$ even, and*
$$Cl_n(\mathbb{C}) = M_{2^n}(\mathbb{C}) \times M_{2^n}(\mathbb{C})$$
*if $n$ is odd.*

To work out the case when the underlying field is $\mathbb{R}$. For any field $k$ we have the following isomorphisms.

**Lemma 0.5.** *For any field $k$*

*(i)* $Cl_{n+2,0}(k) \cong Cl_{0,n}(k) \otimes Cl_{2,0}(k)$.
*(ii)* $Cl_{o,n+2}(k) \cong Cl_{n,0}(k) \otimes Cl_{0,2}(k)$.

*Proof.* Let $e_i$ denote the standard basis of $k^{n+2}$ and cannonical generatoring set of the *Clifford algebra* $Cl_{n+2,0}(k)$.

(i) To get the first isomorphism we simply produce a map given by sending
$$e_i \mapsto e_i \otimes e_i e_2 \forall 1 \leq i \leq n$$
and
$$e_{n+1} \mapsto 1 \otimes e_1$$
$$e_{n+2} \mapsto 1 \otimes e_2$$
It is easy to check that the above map is an isomorphism.

(ii) is similar to $(i)$.

□

One can explicitly check some of the lower dimension cases( $n = 0, 1, 2$). Then one can repeatedly use the isomorphisms in previous lemma. One has to work upto dimension 8 when $k = \mathbb{R}$, before one sees the patern, which is called the *Bott periodicity*. TSome of the calculations are as follows calculations are as follows

(i) $Cl_{0,0}(\mathbb{R}) \cong \mathbb{R}$
(ii) $Cl_{1,0}(\mathbb{R}) \cong \mathbb{C}$
(iii) $Cl_{0,1}(\mathbb{R}) \cong \mathbb{R} \times \mathbb{R}$
(iv) $Cl_{2,0}(\mathbb{R}) \cong \mathbb{H}$
(v) $Cl_{0,2}(\mathbb{R}) \cong M_2(\mathbb{R})$
(vi) $Cl_{3,0}(\mathbb{R}) \cong Cl_{0,1}(\mathbb{R}) \otimes Cl_{2,0}(\mathbb{R}) \cong (\mathbb{R} \times \mathbb{R}) \otimes \mathbb{H} \cong \mathbb{H} \times \mathbb{H}$

(vii) $Cl_{4,0}(\mathbb{R}) \cong Cl_{0,2}(\mathbb{R}) \otimes Cl_{2,0}(\mathbb{R}) \cong M_2(\mathbb{R}) \otimes \mathbb{H} \cong M_2(\mathbb{H})$

(viii) In general one gets,
$Cl_{n+4,0}(\mathbb{R}) \cong Cl_{0,n+2}(\mathbb{R}) \otimes Cl_{2,0} \cong Cl_{n,0}(\mathbb{R}) \otimes Cl_{0,2}(\mathbb{R}) \otimes Cl_{2,0}(\mathbb{R})$

Putting all these observations together we get

**Theorem 0.6.** *The Bott periodicity in case of real number looks like*

$$Cl_{8k+r}(\mathbb{R}) = \begin{cases} M_{2^{4k}}(\mathbb{R}) & r = 0 \\ M_{2^{4k}}(\mathbb{C}) & r = 1 \\ M_{2^{4k}}(\mathbb{H}) & r = 2 \\ M_{2^{4k}}(\mathbb{H}) \times M_{2^{4k}}(\mathbb{H}) & r = 3 \\ M_{2^{4k+1}}(\mathbb{H}) & r = 4 \\ M_{2^{4k+2}}(\mathbb{C}) & r = 5 \\ M_{2^{4k+3}}(\mathbb{R}) & r = 6 \\ M_{2^{4k+3}}(\mathbb{R}) \times M_{2^{4k+3}}(\mathbb{R}) & r = 7 \end{cases}$$

**Lemma 0.7.** *As $k$-algebras $Cl_n^0(k) \cong Cl_{n-1}(k)$*

*Proof.* The isomorphism is given explicitly by the map induced by sending

$$e_i \mapsto e_i.e_n$$

Easy to check that this is an isomorphism of algebras. □